

Finite Time Stability by solving LMIs Problem: Application on four tanks system

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Abstract— In this paper, Finite Time Stability and stabilization of linear continuous-time systems described in the state space are considered. First, we provide a condition for finite time stability using the norm of the state transition matrix. Then we give conditions for the design of a state or static output feedback controllers which finite time stabilize the system. A concrete example illustrates the interest of the proposed approach.

Index Terms—Finite time stability, state feedback, output feedback, Linear Matrix Inequality, four tanks system.

I. INTRODUCTION

When dealing with stability of linear systems, one can refer to numerous works of the literature such as Routh, Hurwitz tests which often necessary and sufficient conditions to analyze stability. However, in some cases, they can present some limitations when applied to some practical problems of interest. Indeed, from an engineering point of view, one is often concerned with the behavior of a system during a transient or a finite time interval which can not be easily captured when analyzing stability by conventional methods. In these cases, a concept which takes into account constraints on the transient behavior is the so-called finite time stability concept, see [1], [2], [4], [6], [10] and references therein. The application of such a concept is justified, for example, when a linear model is obtained from a linearization of a non linear one around an equilibrium point. In fact, the design of an appropriate controller to stabilize the linearized model may lead to unsatisfactory results if the state is not maintained close the linearization point by restricting state excursions. If no constraint on the norm of the state is considered, the control signal may excite the neglected nonlinearities leading to large deviations from the desired objective and in the most dramatic situation to unstable behaviors. Another case of interest concerns the systems controlled by limited actuators. If the state excursions are large, they induce saturations whose effects affect directly closed-loop stability. This paper deals with finite-time stabilization of time-invariant linear systems. A necessary and sufficient condition for finite-time stability analysis expressed through an inequality the norm of the state transition matrix has to satisfy is proposed. It is possible to

deduce a sufficient condition expressed as a linear matrix inequality (LMI). This condition can be used to design a state or static output controls which stabilize the system in the context of finite-time stability. In some cases, it is likely to stabilize asymptotically the system while controlling its motion during transient. In case, it is possible to combine finite-time stabilization with asymptotic stabilization and sufficient conditions expressed through LMIs are proposed. To end, the case of output feedback controller is also studied.

II. PRELIMINARIES

Consider the system described by:

$$\dot{x} = A x \quad (1)$$

Where $A \in \mathbb{R}^{n \times n}$. The following definition can be introduced [3], [7].

Definition 1 System (1) is finite-time stable with respect to $(S_I, S_A, [t_0, t_0 + T])$ if:

$$x(t_0) \in S_I \Rightarrow x(t) \in S_A \quad \forall t \in [t_0, t_0 + T]$$

where S_I is the set of initial states and S_A the set of admissible ones.

This definition is stated in a general way. We can particularize sets S_I and S_A . Interesting classes of sets are represented by ellipsoids. In that case, we have:

$$S_I = \{x \in \mathbb{R}^n : |x| \leq c_1, c_1 > 0\}$$

$$S_A = \{x \in \mathbb{R}^n : |x| \leq c_2, c_2 > c_1 > 0\}$$

For the sequel, we consider definition 1 with the previous definitions for the sets S_I and S_A : System (1) has a unique solution of the form:

$$x(t) = \Phi(t, t_0)x(t_0) \quad (2)$$

where the transition matrix is written as:

$$\Phi(t, t_0) = e^{A(t-t_0)} \quad (3)$$

The following result can be seen as a particularization of the results obtained in [10].

Theorem 1 System (1) is finite-time stable with respect to (c_1, c_2, T) if and only if for all $t \in [t_0, t_0 + T]$

$$\|e^{A(t-t_0)}\| \leq \frac{c_2}{c_1} \quad (4)$$

Which is also equivalent to

$$\|e^{A'(t-t_0)}\| \leq \frac{c_2}{c_1} \quad (5)$$

Proof. The sufficiency follows by the fact that $|Ax| \leq \|A\| \cdot |x|$ and $\|e^A\| \leq e^{\|A\|}$. Indeed $x(t) = |\Phi(t, t_0)x(t_0)| \leq \|e^{A(t-t_0)}\| \cdot |x_0| \leq \|e^{A(t-t_0)}\| c_1$. If the condition of Theorem is satisfied, then $|x(t)| \leq c_2$. The necessity can be proved by contradiction similarly as in [10]. The equivalence with (5) is obtained remarking that (4) is equivalent to

$$\begin{pmatrix} \frac{c_2}{c_1} I & e^{A(t-t_0)} \\ c_1 & I \end{pmatrix} \geq 0$$

It is also possible to obtain another characterization through the solution of a differential Lyapunov equation.

Corollary 1 System (1) is finite-time stable with respect to (c_1, c_2, T) if and only if

$$\lambda_{\max}(P(t)) \leq \frac{c_2}{c_1} \text{ for all } t \in [t_0, t_0 + T] \quad (6)$$

where $P(t)$ satisfies the following differential matrix equation

$$\frac{dP}{dt} = A'P + PA, \quad P(t_0) = I$$

Proof. Defining $P(t) = e^{A'(t-t_0)} e^{A(t-t_0)}$ the proof follows.

Even if they give necessary and sufficient conditions, the previous results are not easy to verify from a computational point of view, particularly the result stated in theorem 1. It is possible to obtain more tractable conditions which are only sufficient.

Theorem 2 If there exist a positive definite symmetric matrix P and a positive scalar β such that

$$0 \leq \beta T \leq \ln\left(\frac{c_2}{c_1}\right) \quad (7)$$

and satisfying the following matrix inequalities

$$AP + PA' - \beta I \leq 0 \quad (8)$$

$$I \leq P \leq \frac{c_2}{c_1} e^{-\beta T} I \quad (9)$$

then system (1) is finite-time stable with respect to (c_1, c_2, T) .

Proof. Suppose that conditions of theorem are satisfied.

Differentiating $e^{A(t-t_0)} P e^{A'(t-t_0)}$ with respect to time leads to

$$\begin{aligned} \frac{d\left(e^{A(t-t_0)} P e^{A'(t-t_0)}\right)}{dt} &= e^{A(t-t_0)} (AP + PA') e^{A'(t-t_0)} \\ &\leq \beta e^{A(t-t_0)} e^{A'(t-t_0)} \leq \frac{\beta e^{A(t-t_0)} P e^{A'(t-t_0)}}{\lambda_{\min}(P)} \end{aligned}$$

By Gronwall.s [11] Lemma we obtain

$$e^{A(t-t_0)} P e^{A'(t-t_0)} \leq e^{\frac{\beta}{\lambda_{\min}(P)}(t-t_0)} I \leq e^{\beta(t-t_0)} I, \quad t \geq t_0 \text{ by (5)}$$

We also have

$$\lambda_{\min}(P) e^{A(t-t_0)} e^{A'(t-t_0)} \leq e^{A(t-t_0)} P e^{A'(t-t_0)} \leq \lambda_{\max}(P) e^{A(t-t_0)} e^{A'(t-t_0)}$$

and then

$$\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)} e^{A(t-t_0)} e^{A'(t-t_0)} \leq e^{A(t-t_0)} P e^{A'(t-t_0)} \leq e^{\beta(t-t_0)} I, \quad t \geq t_0$$

We deduce that

$$\|e^{A'(t-t_0)}\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} e^{\beta(t-t_0)}, \quad t > t_0$$

And

$$\|e^{A'(t-t_0)}\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} e^{\beta T} \leq \frac{c_2^2}{c_1^2} \text{ for all } t \in [t_0, t_0 + T] \text{ by (6)}$$

which conclude the proof.

From the previous theorem, we can deduce the following interesting result.

Corollary 2 The conditions of theorem 1 are satisfied if there exists a positive scalar γ satisfying

$$A + A' - \gamma I \leq 0 \quad (10)$$

$$\gamma T \leq 2 \ln\left(\frac{c_2}{c_1}\right) \quad (11)$$

and then system (1) is finite-time stable with respect to (c_1, c_2, T) . In addition, for all values of c_1, c_2 and T satisfying

$$2 \ln\left(\frac{c_2}{c_1}\right) \geq T \lambda_{\max}(A + A') \quad (12)$$

system (1) is finite-time stable with respect to (c_1, c_2, T) .

Proof. Suppose that conditions of corollary are satisfied. Then $P=I$ and $\beta=\gamma$ satisfy conditions of theorem 2. In addition if the first inequality is satisfied we have

$$\gamma I \geq \lambda_{\max}(A+A')$$

and the last part of corollary follows.

We can state the following result for the case of asymptotically stable systems.

Corollary 3 If there exist a positive definite symmetric matrix P satisfying the following linear matrix inequalities (LMIs)

$$AP+PA' < 0 \quad (13)$$

$$I \leq P \leq \frac{c_2^2}{c_1^2} I \quad (14)$$

then system (1) is asymptotically and finite-time stable with respect to (c_1, c_2, T) .

Proof. Suppose that conditions of corollary are satisfied. Following the proof of theorem 1, we obtain

$$\|e^{A(t-t_0)}\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} e^{\beta(t-t_0)}, \quad t > t_0$$

But now, we have because $\beta=0$

$$\|e^{A(t-t_0)}\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \leq \frac{c_2^2}{c_1^2} \text{ for all } t \in [t_0, t_0 + T]$$

which conclude the proof

III. FINITE TIME STABILIZATION

We consider in this section the following system

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \quad (15)$$

Where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. The problem addressed in this paragraph can be stated as follows

Problem 1 Find a state feedback control $u = Kx$ with $K \in \mathbb{R}^{m \times n}$ such that system (15) is practically stable with respect to (c_1, c_2, T) .

From theorem 2, it is possible to deduce the following result.

Theorem 3 If there exist a positive definite symmetric matrix S ; a matrix R of appropriate dimension and a positive scalar β such that

$$0 \leq \beta T \leq \ln\left(\frac{c_2}{c_1}\right) \quad (16)$$

and satisfying the following matrix inequalities

$$AS + S'A' + BR + R'B' - \beta I \leq 0 \quad (17)$$

$$I \leq S \leq \frac{c_2^2}{c_1^2} e^{-\beta T} I \quad (18)$$

then the control law $u = RS^{-1}x$ solves Problem 1.

Proof. The proof follows from the fact that second inequality can be written

$$(A + BRS^{-1})S + S(A + BRS^{-1})' - \beta I \leq 0$$

It is also possible to obtain directly a control law from the extension of corollary

Corollary 4 If there exist a positive scalar γ , a matrix K of appropriate dimensions satisfying

$$A + BK + A' + K'B' - \gamma I \leq 0 \quad (19)$$

$$\gamma T \leq 2 \ln\left(\frac{c_2}{c_1}\right) \quad (20)$$

Then the control law $u = Kx$ solve problem

IV. APPLICATION ON FOUR TANKS SYSTEMS

In this application, we treat the case of stabilization in finite time using the LMI method on four tanks system. The considered system is nonlinear that's why, when using a linear order. It guarantees that the state is bounded near the operating point. The purpose of this application is to synthesize a controller to finite time stabilize the system.

A. System modeling

To simplify calculations, we will assimilate the four tanks system to that given by the block diagram (Fig 1). We will do a full study of the system to achieve equations that govern it.

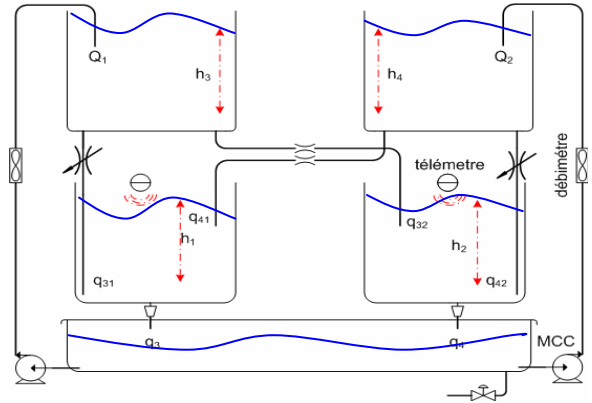


Fig 1 - Four tanks system synoptic schematic.

The study could be made in the case of a single tank then it will be generalized to the entire installation set. Considering the tank of the figure 2.

- with:
- q_e : in flow
 - q_s : out flow
 - h : water level in the tank
 - S : bottom surface of the tank
 - s : section of the outlet opening
 - V_e : velocity of the fluid at the input
 - V_s : velocity of the fluid at the output

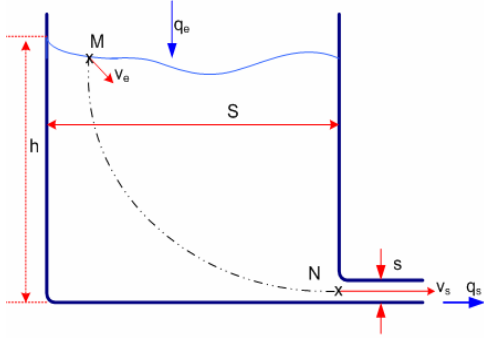


Fig 2 - Elementary tank

the tank contains a liquid mass m

$$m = \rho.V = \rho.S.h \quad (21)$$

where ρ is the density (in kg/m³).

According to the principle of conservation of the mechanical energy, for each point M located at a height h of the orifice

$$\frac{1}{2}m.V_e^2 + \rho.g.S.h^2 = cste \quad (22)$$

As $S \gg s$, $V_e \cong 0$, and then :

$$\rho.g.S.h^2 = cste \quad (23)$$

Same for each point N located at the base of the tank we have:

$$\frac{1}{2}m.V_s^2 = cste \quad (24)$$

If we accept that there is an equality between the relation (23) and (24), we find the Bernoulli distribution:

$$V_s = \sqrt{2gh} \quad (25)$$

The flow's variation is giving by

$$S \frac{dh}{dt} = (q_e - q_s) \quad (26)$$

from (25), we have

$$\frac{dh}{dt} = \frac{1}{S} q_e - \frac{s}{S} \sqrt{2gh} \quad (27)$$

Generalization to tanks set:

The generalization of equation (27) for a tanks set, we can obtain the following relations:

$$\begin{cases} S \frac{dh_1}{dt} = q_{31} + q_{41} - q_3 \\ S \frac{dh_2}{dt} = q_{32} + q_{42} - q_4 \\ S \frac{dh_3}{dt} = Q_1 - q_{31} - q_{32} \\ S \frac{dh_4}{dt} = Q_2 - q_{41} - q_{42} \end{cases} \quad (28)$$

with :

- q_{ij} is the flow of the tank i in the tank j
- $q_{ij} = s_{ij} \sqrt{2gh_i}$, $i \in \{3,4\}$, $j \in \{1,2\}$
- Q_k flow of the pump k , $k \in \{1,2\}$

Then we replace q_{ij} in (28) which gives :

$$\begin{cases} \frac{dh_1}{dt} = -\frac{s_3}{S} \sqrt{2gh_1} + \frac{s_{31}}{S} \sqrt{2gh_3} + \frac{s_{41}}{S} \sqrt{2gh_4} \\ \frac{dh_2}{dt} = -\frac{s_4}{S} \sqrt{2gh_2} + \frac{s_{32}}{S} \sqrt{2gh_3} + \frac{s_{42}}{S} \sqrt{2gh_4} \\ \frac{dh_3}{dt} = -\frac{(s_{31} + s_{32})}{S} \sqrt{2gh_3} + \frac{1}{S} Q_1 \\ \frac{dh_4}{dt} = -\frac{(s_{41} + s_{42})}{S} \sqrt{2gh_4} + \frac{1}{S} Q_2 \end{cases} \quad (29)$$

To simplify the notations, rewrite equation (29) as follows:

$$\begin{cases} \dot{h}_1 = -c_1 \sqrt{h_1} + c_2 \sqrt{h_3} + c_3 \sqrt{h_4} \\ \dot{h}_2 = -c_4 \sqrt{h_2} + c_5 \sqrt{h_3} + c_6 \sqrt{h_4} \\ \dot{h}_3 = -c_7 \sqrt{h_3} + c_8 U_1 \\ \dot{h}_4 = -c_9 \sqrt{h_4} + c_{10} U_2 \end{cases} \quad (30)$$

where $c_i, i \in \{1, \dots, 10\}$ are the system's constant.

The values of the numerical parameters for the four tanks are given in Table 1

Constants	Numerical values
c_1, c_4	0,0119
c_2, c_6	0,014951
c_3, c_5	0,002
c_7, c_9	0,020333
c_8, c_{10}	4,3

Table 1 - Four tanks system constants

B. Linearization around an operating point

It is a system of nonlinear equations, we seek to linearize and determine a state model.

That's why, we fix an operating point $Q_{10}, Q_{20}, h_{10}, h_{20}, h_{30}, h_{40}$, knowing that levels depend on flows, they therefore can not be set arbitrarily in applying digital will treated as a last resort.

We set $h_i = H_i + H_{i0}$ and $Q_i = U_i + q_{i0}$ where H_i and U_i represent variations around the operating point.

Then replaced in (30), we obtain:

$$\begin{cases} \frac{d(H_1 + h_{10})}{dt} = -\frac{s_3}{S} \sqrt{2g(H_1 + h_{10})} + \frac{s_{31}}{S} \sqrt{2g(H_3 + h_{30})} \\ \quad + \frac{s_{41}}{S} \sqrt{2g(H_4 + h_{40})} \\ \frac{d(H_2 + h_{20})}{dt} = -\frac{s_4}{S} \sqrt{2g(H_2 + h_{20})} + \frac{s_{32}}{S} \sqrt{2g(H_3 + h_{30})} \\ \quad + \frac{s_{42}}{S} \sqrt{2g(H_4 + h_{40})} \\ \frac{d(H_3 + h_{30})}{dt} = -\frac{(s_{31} + s_{32})}{S} \sqrt{2g(H_3 + h_{30})} + \frac{1}{S}(U_1 + Q_{10}) \\ \frac{d(H_4 + h_{40})}{dt} = -\frac{(s_{41} + s_{42})}{S} \sqrt{2g(H_4 + h_{40})} + \frac{1}{S}(U_2 + Q_{20}) \end{cases} \quad (31)$$

To simplify the previous expression and eliminate the square roots of the terms H_1, H_2, H_3 and H_4 , we use the following Taylor expansion:

$$\sqrt{1+x} = 1 + \frac{x}{2} \quad \text{when } x \ll 1 \quad (32)$$

More so, $h_{i0} = cste$ then $\frac{dh_{i0}}{dt} = 0$, after simplification we get:

$$\begin{cases} \frac{dH_1}{dt} = -\frac{s_3}{S} \sqrt{\frac{g}{2h_{10}}} H_1 + \frac{s_{31}}{S} \sqrt{\frac{g}{2h_{30}}} H_3 + \frac{s_{41}}{S} \sqrt{\frac{g}{2h_{40}}} H_4 \\ \frac{dH_2}{dt} = -\frac{s_4}{S} \sqrt{\frac{g}{2h_{20}}} H_2 + \frac{s_{32}}{S} \sqrt{\frac{g}{2h_{30}}} H_3 + \frac{s_{42}}{S} \sqrt{\frac{g}{2h_{40}}} H_4 \\ \frac{dH_3}{dt} = -\frac{(s_{31} + s_{32})}{S} \sqrt{\frac{g}{2h_{30}}} H_3 + \frac{1}{S} U_1 \\ \frac{dH_4}{dt} = -\frac{(s_{41} + s_{42})}{S} \sqrt{\frac{g}{2h_{40}}} H_4 + \frac{1}{S} U_2 \end{cases} \quad (33)$$

C. Simulation's results

The aim of this application is to maintain the water level in each tank, below a threshold selected by the user during a given time interval. It will substantially stabilize the water level in the four tanks.

We linearize the system around the point:

$$h_0 = [0,3145 \quad 0,3918 \quad 0,15 \quad 0,2]^T \quad (34)$$

We obtain the linear system:

$$\dot{h}(t) = \begin{bmatrix} -0,0106 & 0 & 0,0193 & 0,0022 \\ 0 & -0,0095 & 0,0026 & 0,0167 \\ 0 & 0 & -0,0262 & 0 \\ 0 & 0 & 0 & -0,0227 \end{bmatrix} h(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4,3 & 0 \\ 0 & 4,3 \end{bmatrix} u(t) \quad (35)$$

The system (35) is stable over time compared to $(0,7; 0,8; 200)$ through the corrector

$$K = \begin{pmatrix} -0,0047 & -0,0006 & 0,0205 & 0 \\ -0,0005 & -0,0041 & 0 & -0,0352 \end{pmatrix}$$

from Theorem 3 with

$$R = \begin{pmatrix} -0,0150 & -0,0020 & 0,0628 & 0 \\ -0,0016 & -0,0120 & 0 & -0,1079 \end{pmatrix}$$

and

$$S = \begin{pmatrix} 3,0880 & 0 & -0,0193 & -0,0022 \\ 0 & 3,0859 & -0,0026 & -0,0168 \\ -0,0193 & -0,0026 & 3,0668 & 0 \\ -0,0022 & -0,0168 & 0 & 3,0668 \end{pmatrix}$$

Figure 3 shows the evolution of the state, Figure 4 shows the evolution of the norm of the state and Figure 5 shows the evolution of the two components of the order.

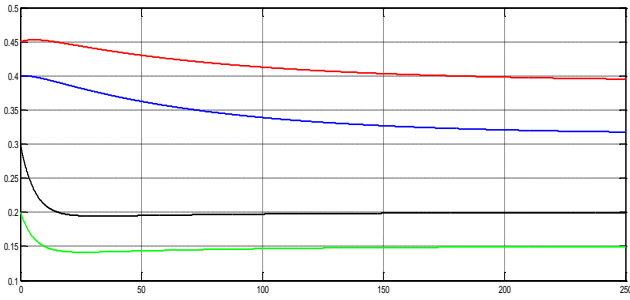


Fig 3 - State Evolution

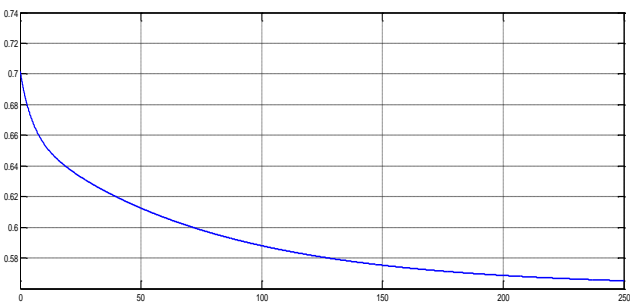


Fig 4 - Norm State Evolution

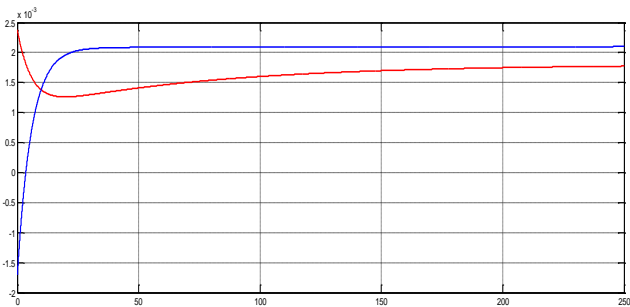


Fig 5 - Control evolution

We can also extend theorem 3 to the case of output feedback controller.

V. EXTENSION ON OUTPUT FEEDBACK

Now if the state is not available and only an output $y = Cx$ is measurable, it is possible to extend the previous corollary to the case of static output feedback one.

Corollary 5 If there exist a positive scalar γ , a matrix L of appropriate dimensions satisfying

$$A + BLC + A' + C'L'B' - \gamma I \leq 0 \quad (36)$$

$$\gamma T \leq 2 \ln \left(\frac{c_2}{c_1} \right) \quad (37)$$

then the control law $u = Ly$ solves Problem 1.

V. CONCLUSION

In this paper, we have considered the finite time control problem for linear system with state feedback. First of all we have extended the definition of FTS into the definition of PS. Then we have provided new sufficient LMI conditions guaranteeing PS and FTS via state feedback. The conditions have been turned into an optimization problem involving LMIs. The result has been extended to the output feedback case. Finally a numerical example shows that the proposed conditions provide useful and less conservative results.

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